An Insight Into Quantum Random Walks

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Introduction

What exactly is a quantum random walk?

- The term was first coined in the 1993 paper by Aharnov, Davidovich and Zaguray[1], although Feynmann[2] had introduced the idea of a quantum walk many years before in 1940, except he did not call it a quantum walk
- There is both a discrete and continuous quantum random walk, just as in the classical case
- They can also be applied to an undirected graph $G(V, E)$, which means we can implement them as quantum algorithms
- The behaviour of both the discrete and continuous quantum random walks is very different from the classical case
- They do not have gaussian properties and they do not converge to limiting distributions, which means it is very hard to analytically calculate their statistical properties
- The width(standard deviation) of both the distributions on the line is found to be $\sigma_{Quant} = t$, which is much greater than the classical case of $\sigma_{Class} = \sqrt{t}$
- Which means the walker travels much further on average from their starting position!
- It is this property of the walks that has sparked a lot of excitement, especially in regards to algorithmic development for quantum computers

Discrete Quantum Walk

The Set Up

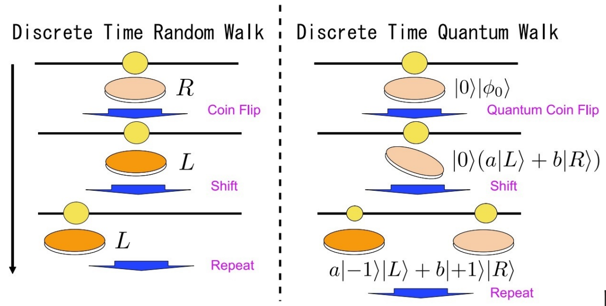
- To implement the walk on the line we introduce a coin space $\mathcal{H}_c$ and a position space $\mathcal{H}_p$, so our total space is $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$, where we must introduce a unitary operator $\hat{U}$ that acts on $\mathcal{H}$
- Where the coin space represents the states of a coin and is spanned by the basis vectors $\{|R\rangle, |L\rangle\}$. The position space is spanned by the position states $\{|x\rangle : x \in \mathbb{Z}^{+,0}\}$
- We first prepare the system in some initial state $|\psi(0)\rangle = |0\rangle \otimes |\phi(0)\rangle$, where $|\phi(0)\rangle = |L\rangle, |R\rangle$ or a superposition of the two
- We next introduce a translational shift operator $\hat{S} : |x+1\rangle \otimes |R\rangle$, which only acts on $\mathcal{H}_p$
- We then introduce our coin operator $\hat{C}$. If we would like the symmetric quantum walk on the line, we can choose $\hat{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$
- This can also be done via a different coin $\hat{C}$ and changing our initial condition $|\phi(0)\rangle$ to generate the same effect.

The Walk

- So the dynamics of the walk is described by the unitary acting on the initial state $|\psi(0)\rangle$

$$|\psi(t+1)\rangle = \hat{U}^t\,|\psi(t)\rangle = (\hat{S}(\hat{C} \otimes \hat{I}))^t\,|\psi(0)\rangle \tag{1}$$

- which leads to the symmetrical, non-Gaussian distribution in figure 2



[3]

Figure 1: A pictorial representation of how the discrete quantum walk evolves. As more iterations are carried out, the "walker" can be in a superposition of multiple position states. This is ultimately why the quantum walker travels so much further.

Continuous Random Walk

Set Up

- The continuous time random walk does not have the extra structure of the coin space $\mathcal{H}_c$, instead its Hilbert space $\mathcal{H}$ is simply the position space $\mathcal{H}_p$
- As in the continuous Markov process, the time is also continuous and takes place on the graph $G(V, E)$, except the vertices are now the quantum states $\{|1\rangle, \ldots, |v\rangle\}$
- We aim to turn our transition matrix M, in to a unitary operator

$$M_{ab} = \begin{cases} k\gamma, & a = b \text{ and } k \text{ is the degree of vertex } a \\ -\gamma, & a \neq b, a \text{ and } b \text{ are connected by an edge} \\ 0, & a \neq b, a \text{ and } b \text{ are not connected} \end{cases} \tag{2}$$

The Walk

- We do this by multiplying the transition matrix by the imaginary unit i [4], which then turns our transition matrix $\hat{M}$ into the Hamiltonian $\langle a|\hat{H}|b\rangle = M_{ab}$
- Solving the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\langle a|\psi(t)\rangle = \sum_b \langle a|\hat{H}|b\rangle\,\langle b|a\rangle \tag{3}$$

for a given Hamiltonian, with initial condition $M_{ab} = \delta_{ab}$
- We gain the following unitary which describes the dynamics of the walk, $\hat{U} = e^{-i\hat{H}t}$
- Thus our walk evolves, for some initial state $|\psi(0)\rangle$ as follows $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$
- For the walk on the line when $k = 2$, we get exactly the same distribution as the discrete case(figure 2), despite their very different constructions

Probability Distributions

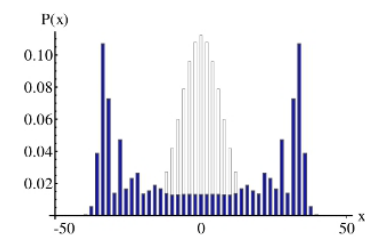


Figure 2: Probability distributions of the discrete and continuous quantum and classical walks on the line, for $t = 50$

Grover's Algorithm

What Is It?

- Originally discovered by Lov Grover in 1996-1997 [5]
- It is a very simple , but powerful database search algorithm that makes use of the linear superposition of states
- Imagine that you have a list of N names , that are unsorted,and you only want one of those names. Grover's algorithm will find the given name "marked state", in a computational time $t = \mathcal{O}(\sqrt{N})$
- This is remarkable, as the equivalent classical algorithm takes a time of $t = \mathcal{O}(N)$! This means that if you had a 1,000,000 unsorted names, Grover's algorithm, on average, would find the name after searching 1,000 names . Whereas the classical case, would on average, have to search through at least 500,000 names!

Set Up

- We begin by preparing all states(elements of our database) in a linear superposition $|\psi\rangle = \sum_{i=1}^{N} \frac{1}{\sqrt{N}}|i\rangle$, (figure 3a) where $\mu = \frac{1}{N}$
- Then we continually apply Grover's unitary operator $\hat{G}$ recursively to an initial state $|\psi(0)\rangle = |\psi\rangle \otimes |-\rangle$ and then make a measurement on the final state $|\psi(t)\rangle$ after a number of steps t. Where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- Grover's operator consists of an oracle $\hat{O}$, which acts as a function that shifts the phase of the marked state by π , whilst leaving the rest unchanged (figure 3b) and a diffusion transform $\hat{D}$, that inverts our marked state about the mean (figure 3c)

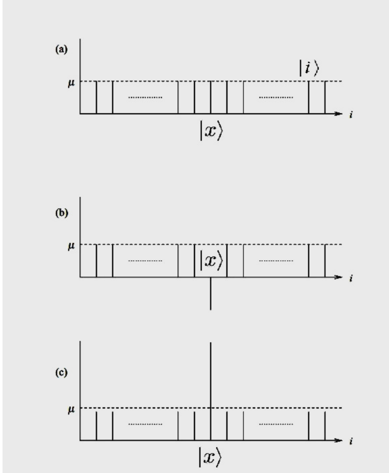


Figure 3: Amplitude amplification and Grover's alogirthm at work

A Modified Version of Grover's Algorithm

- Due to the impressive properties of the algorithm, people had tried to apply Grover's algorithm directly to a spatial search, but unfortunately the algorithm reverts back to a computation time of $\mathcal{O}(N)$ [6]. As in a spatial search there is an additional time cost as we move between memory locations
- Ambainis *et al.* [7] showed that we can use the construction of the discrete quantum walk to give us Grover's algorithm. In doing this they were able to get a computation time of $\mathcal{O}(\sqrt{N}\log N)$ for 2-dimensions and $\mathcal{O}(\sqrt{N})$ for 3 or more dimensions
- In using the walk for this simple case, we find that it not only enhanced computation time, but it also offered a new approach for constructing quantum algorithms

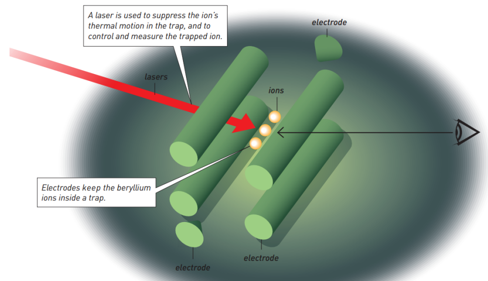
Experimental Implementation

Quantum Computers

- To effectively implement the walks, we must be able to design a quantum computer
- There are several different types of experimental set ups , from nuclear magnetic resonance(NMR), optical photon computer, harmonic oscillator computer, ion traps and many others [8]
- We specifically focus on ion trapping, due to it's similarities to the discrete walk and recent successes [9]

Trapped Ion Experiment

- We begin with a single Beryllium ion Be^{+} confined in a coaxial resonator radio frequency ion trap
- Next we create the unitary equation(1), by applying a sequence of four Raman beam pulses, to create a superposition state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle|\downarrow\rangle + |-\alpha\rangle|\uparrow\rangle)$, where $|\alpha\rangle$ are the coherent states
- Then we measure the internal state of the ion, with the measurement operator $\hat{M} = e^{\pm i\hat{p}\sigma_2}$, where $i\hat{p}\sigma_2$ is the Hamiltonian, which Hamiltonian we use is dependent on the internal state of the system. We then measure the internal state again
- If decoherence has effected the ion, we revert back to the classical walk. If there is no decoherence, then we get the quantum walk.



[10]

Figure 4: Ion trap. Using electromagnetic fields and light to confine, control, and measure the quantum state of beryllium ions.

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